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q-Deformed Probability and Binomial Distribution

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We introduce the concept of q-deformed probability and discuss the q-deformed binomial distribution.

1. INTRODUCTION

Quantum groups or q-deformed Lie algebras imply some specific deformations of classical Lie algebras. One of the simplest examples of a quantum group is a q-deformed version of the classical $su(2)$ algebra, which is denoted by $su_q(2)$ (Jimbo, 1985; Drinfeld, 1986). A realization of $su_q(2)$ has been obtained by Arik and Coon (1976), Macfarlane (1989), and Biedenharn (1989). Most of them used the Jordan–Schwinger mapping to obtain the $su_q(2)$ algebra in terms of the q-deformed Heisenberg–Weyl algebra, which is also called the q-deformed boson algebra because the algebra goes to the ordinary boson algebra when the deformation parameter goes over to some particular value.

Recently there has been some interest in more general deformations involving arbitrary real functions of weight generators and including q-deformed algebras as a special case (Polychronakos, 1990; Roczek, 1991; Daskaloyannis, 1991; Chung *et al.*, 1993; Chung, 1994).

In statistical physics the concepts of combinatorics and probability are very important. Several types of distributions are especially crucial in developing statistical models. From time to time, a new statistical distribution brings about new physics or a new physical model. Recently Chung and Kang (1994) developed the theory of q-deformed combinatorics by introducing the

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new concept of q-selecting and q-order. In this paper we introduce the concept of the q-deformed probability to construct the q-deformed binomial distribution.

2. THEORY OF q-DEFORMED PROBABILITY

In this section we introduce the concept of q-deformed probability which we call q-probability. Using this, we define a q-deformed binomial experiment. To begin with we impose the following conditions on the q-deformed binomial experiment.

1. There must be a fixed number of trials, which we denote by N.
2. Each trial must result in a "success" or a "failure."
3. Trials are no longer independent. Instead we impose the following rules:

3.1. The probability that the event will occur when the previous event occurs is equal to the probability that the previous event occurs.

3.2. The probability that the event will occur when the previous event does not occur is equal to q times the probability that the previous event occurs.

4. In each trial, the sum of the probability of a success and that of a failure is equal to unity.

Now let us consider the case that the number of trials is three. Let S and F be a success and a failure, respectively. Since trials are not independent in the q-deformed binomial experiment, we stress that the order of outcomes is crucial. Consider the q-deformed sample space of possible outcomes for three trials. Let the probability of S be p. Then, in the first trial, we have

$$P_q(S) = p, \qquad P_q(F) = 1 - p$$

where P_q means the q-deformed probability. Since the outcomes on the three trials are not independent, the q-probability assigned to any one of the sample points is obtained by multiplying three q-probabilities which satisfy the rules of the deformed probability. For instance, let us consider the case that we get a success in the first trial and a failure in the second and third trials, which will be denoted by SFF. Then the probability of SFF is given by

$$P_q(SFF) = p \times (1 - p) \times (1 - qp)$$

where the q-factor in the last term appears because the previous event is a failure, so the probability of S is q-deformed. Following the rule of the q-deformed binomial experiment, we can obtain the sample space of possible outcomes for three trials; see Table I. From the q-deformed sample space given in Table I we can build the q-probability table in the q-deformed binomial distribution as shown in Table II.

Table I.

Sample space	q-Probability
SSS	$p \times p \times p$
SSF	$p \times p \times (1 - p)$
SFS	$p \times (1 - p) \times qp$
FSS	$(1 - p) \times qp \times qp$
SFF	$p \times (1 - p) \times (1 - qp)$
FSF	$(1 - p) \times qp \times (1 - qp)$
FFS	$(1 - p) \times (1 - qp) \times q^2 p$
FFF	$(1 - p) \times (1 - qp) \times (1 - q^2 p)$

Table II.

Number of S	q-Probability
0	$(1 - p)(1 - qp)(1 - q^2 p)$
1	$[3]p(1 - p)(1 - qp)$
2	$[3]p^2(1 - p)$
3	p^3

Here the q-number $[n]$ is defined as

$$[n] = \frac{q^n - 1}{q - 1}$$

From Table II the q-probability of getting n S's in three trials is given by

$$P_3^q(n) = {}_3C_n^q p^n (1 - p)_q^{3-n}$$

where the q-binomial is defined as

$$(x + y)_q^n = \prod_{k=0}^{n-1} (x + q^k y)$$

and

$$ {}_nC_k^q = \frac{[n]!}{[n - k]![k]!}, \qquad [n] = \frac{1 - q^n}{1 - q}$$

Simple computation shows that sum of each q-probability is 1.

3. THE q-DEFORMED BINOMIAL DISTRIBUTION

In this section we obtain the general formula for the q-deformed binomial distribution. Now let us consider the case that there exist r successes in N

trials. Since the position of a success is important in the q-deformed binomial distribution, the positions of r successes are written in the sequence

$$\{i_1, i_2, \ldots, i_r\}$$

where we arrange the r positions according to the rule

$$i_1 < i_2 < \cdots < i_r$$

Then the q-probability that successes will appear in i_1th, i_2th, ..., i_rth trials is

$$P_N(i_1, i_2, \ldots, i_r) = \prod_{k=0}^{N-(r+1)} (1 - q^k p) \times \prod_{k=1}^{r} q^{i_k - k} p$$
$$= q^{\sum_{k=1}^{r} i_k - r(r+1)/2} p^r (1 - p)_q^{N-r} \tag{1}$$

Therefore the q-probability of r successes among N trials is

$$P_N^q(n) = \sum_{i_1<i_2<\cdots<i_r} q^{\sum_{k=1}^{r} i_k - r(r+1)/2} p^r (1 - p)_q^{N-r}$$
$$= {}_N C_r^q p^r (1 - p)_q^{N-r} \tag{2}$$

where we used the useful identity

$$\sum_{i_1<i_2<\cdots<i_r} q^{\sum_{k=1}^{r} i_k - r(r+1)/2} = {}_N C_r^q \tag{3}$$

The proof of equation (3) is given in Appendix A.

From the definition of the q-binomial and the properties of the q-deformed binomial coefficient, we can easily show that the sum of all the q-probabilities in the q-deformed binomial experiment is unity, which means that

$$\sum_{r=0}^{N} P_N^q(n) = \sum_{r=0}^{N} {}_N C_r^q p^r (1 - p)_q^{N-r} = 1 \tag{4}$$

The proof of identity (4) is given in Appendix B.

Using the definition of the q-binomial distribution, we can easily show that the q-expectation value and q-root mean square deviation are written as

$$\langle [r] \rangle_q = \sum_{r=0}^{N} [r] P_N^q(r) = [N]p \tag{5}$$

and

$$\sigma_q = \{<([r] - \langle [r] \rangle_q)^2 \rangle_q\}^{1/2} = \{[N]p(1 - p)\}^{1/2} \tag{6}$$

4. CONCLUSION

In this paper we introduced the new concepts of the q-deformed probability and the q-deformed binomial experiment. We use these results to construct the q-deformed binomial distribution. We think that this type of new distribution may be related to some new physical models whose hidden symmetry is a quantum group. In addition, we suppose that the q-deformed binomial distribution is connected with q-deformed statistical physics. We hope that these problems and related topics will become clear in the near future.

APPENDIX A. PROOF OF EQUATION (3)

In this appendix we prove the identity (3):

$$
\begin{aligned}
{}_N C_r^q &= \sum_{i_1<i_2<\cdots<i_r} q^{i_1-1} q^{i_2-2} \cdots q^{i_r-r} \\
&= \sum_{i_1<i_2<\cdots<i_r} q^{\sum_{k=1}^r i_k - r(r+1)/2}
\end{aligned}
$$

In order to use mathematical induction, we assume that the above relation holds for N trials. For $N + 1$ trials, the RHS of equation (3) is given by

$$K = \sum_{I_1<I_2<\cdots<I_r} q^{\sum_{j=1}^r I_j - r(r+1)/2}$$

where

$$\{I_1, I_2, \ldots, I_r\} \subset \{1, 2, \ldots, N, N + 1\}$$

Then we have

$$K = \sum_{i_1<i_2<\cdots<i_r} q^{\sum_{j=1}^r i_j - r(r+1)/2} + \sum_{i_1<i_2<\cdots<i_{r-1}<i_r=N+1} q^{\sum_{j=1}^{r-1} i_j + N + 1 - r(r+1)/2}$$

where

$$\{i_1, i_2, \ldots, i_{r-1}, i_r\} \subset \{1, 2, \ldots, N\}$$

Hence we have

$$K = {}_N C_r^q + q^{N+1-r}\, {}_N C_{r-1}^q = {}_{N+1} C_r^q$$

which completes the proof of Equation (3) according to the induction principle.

APPENDIX B. PROOF OF EQUATION (4)

In this appendix we prove the relation (4),

$$S_N = \sum_{r=0}^{N} {}_N C_r^q p^{N-r} (1 - p)_q^r = 1$$

From equation (4) we have

$$S_N = {}_NC_0^q p^N + \sum_{r=1}^{N-1} {}_NC_r^q p^{N-r}(1-p)_q^r + {}_NC_N^q(1-p)_q^N \tag{B1}$$

Then the second term of RHS of the above equation is written in the form

$$\sum_{r=1}^{N-1} {}_NC_r^q p^{N-r}(1-p)_q^r$$
$$= \sum_{r=1}^{N-1} q_{N-1}^r C_r^q p^{N-r}(1-p)_q^r + \sum_{r=1}^{N-1} {}_{N-1}C_{r-1}^q p^{N-r}(1-p)_q^r \tag{B2}$$

where we used the formula

$${}_NC_r^q = q_{N-1}^r C_r^q + {}_{N-1}C_{r-1}^q$$

After some computation we get

$$S_N = \sum_{r=0}^{N-1} q_{N-1}^r C_r^q p^{N-r-1}(1-p)_q^r p + \sum_{r=0}^{N-1} {}_{N-1}C_r^q p^{N-1-r}(1-p)_q^r(1-q^r p)$$
$$= \sum_{r=0}^{N-1} {}_{N-1}C_r^q p^{N-1-r}(1-p)_q^r$$
$$= S_{N-1} \tag{B3}$$

which completes the proof of relation (4) by mathematical induction.

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